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Directed Paths on Percolation Clusters

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We consider a variant of the problem of directed polymers on a disordered lattice, in which the disorder is "geometrical" in nature. In particular, we allow a finite probability for each bond to be absent from the lattice. We show, through the use of numerical and scaling arguments on both Euclidean and hierarchical lattices, that the model has two distinct scaling behaviors, depending upon whether the concentration of bonds on the lattice is at or above the directed percolation threshold. We are particularly interested in the exponents ω and ζ, defined by $\delta f \sim t^{\omega}$ and $\delta x \sim t^{\zeta}$, describing the free-energy and transverse fluctuations, respectively. Above the percolation threshold, the scaling behavior is governed by the standard "random energy" exponents ($\omega = 1/3$ and $\zeta = 2/3$ in $1+1$ dimensions). At the percolation threshold, we predict (and verify numerically in $1+1$ dimensions) the exponents $\omega = 1/2$ and $\zeta = \nu_{\perp}/\nu_{||}$, where $\nu_{\perp}$ and $\nu_{||}$ are the directed percolation exponents. In addition, we predict the absence of a "free phase" in any dimension at the percolation threshold.

KEY WORDS: Directed polymers; percolation; random walks; hierarchical lattices; disorder.

1. INTRODUCTION

Recently, there has been much interest in the problem of directed paths in a random potential[1-3] (see ref. 2 for a recent review). In addition, there has been a general interest in a large class of problems involving directed paths and closely related questions in interface growth.[4] Here we consider a similar problem with the "geometrical" disorder associated with the voids of a percolation cluster. Such percolation clusters have been frequently used as models for disordered materials.[5] This differs from the case of a (directed) path in a random potential only in that the weight for each bond

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has a finite probability of being zero as well as positive. The allowed configurations of such paths have been used to model cracks formed in ductile fracture of a porous plate.[6] The mapping to interface growth[1,4,7] also relates this to the evolution of a poisoned Eden model[8,9] in which the addition of "poison" particles permanently stops the growth around a particular site. Many related growth models have been studied, a number of which are also related to directed percolation.[10–13]

The simplest version of such a problem consists of a hypercubic lattice, in which each bond is present (with weight 1) with probability p and absent with probability $1-p$. We consider walks oriented along the diagonal of such a lattice, fixed at one end with the other end free (see Fig. 1). These paths are constrained to walk only on the sites which are present in the particular realization of the diluted lattice. Averages are calculated by summing over all such walks, with a weight that may include a random potential on each of the bonds, or may simply be an unweighted sum over all possible paths. For each point $(\mathbf{x}, t)$ on the lattice (see Fig. 1), we can define an overall weight $W(\mathbf{x}, t)$ of all possible paths connecting it to the origin. These weights can then be used to compute typical transverse fluctuations $\delta x(t)$ and free energies $f(\mathbf{x}, t) = \ln W(\mathbf{x}, t)$. The (universal) nature of the fluctuations of these quantities is captured by the two exponents ω and ζ characterizing their scaling with the path length t as

$$\begin{aligned} \delta f &\sim t^{\omega} \\ \delta x &\sim t^{\zeta} \end{aligned} \tag{1.1}$$

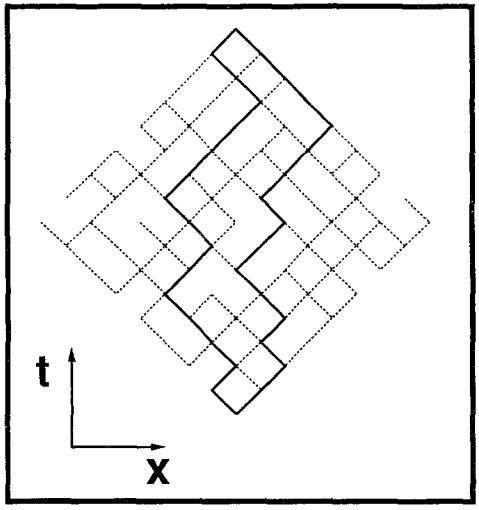

Fig. 1. Directed paths (solid lines) on a diluted 2D lattice (dashed bonds).

For a percolation problem, because the free energy and average displacement are undefined for some (disconnected) realizations of the randomness, it is necessary to define the above quantities restricted to the connected clusters. Above the percolation threshold the disconnected clusters have infinitesimal weight for large lattices, so one expects to recover the random potential results. Below the (directed) percolation threshold, the connected configurations have vanishing weight, so the quantities are not very well defined. It is interesting to consider what happens precisely at the percolation threshold. In particular, we consider the values of the exponents ω and ζ both above and at the transition. Note that since the walks are assumed to be directed, the transition of interest is that of *directed percolation*, which occurs for a value of p above the usual percolation threshold.

We find that, indeed, above the percolation threshold, the fluctuation exponents take on their usual random-bond values ($\omega = 1/3$ and $\zeta = 2/3$ in $1+1$ dimensions). At the percolation threshold, we find that the free energy fluctuations follow the one-dimensional scaling $\omega = 1/2$, while the transverse fluctuations are limited by the anisotropic fractal shape of the percolation cluster[5] $\zeta = \nu_\perp/\nu_\parallel$. In the usual random-bond case, there is a "free" phase[1,14] for sufficiently weak randomness and high dimensions, where randomness is irrelevant ($\omega = 0$ and $\zeta = 1/2$). Since we expect the result $\omega = 1/2$ to hold at the percolation threshold in all dimensions, we find that such a "free" phase is never present at the transition. These conclusions are based on results on hierarchical lattices (Section 2) and numerical simulations on two-dimensional square lattices (Section 3).

2. HIERARCHICAL LATTICES

As a first step, we consider the problem on the hierarchical lattices indicated in Fig. 2. The lattice is constructed iteratively, by replacing each bond in the lattice at the nth stage, L_n, by $2b$ bonds to create the lattice at the next iteration, L_{n+1}. Such lattices were introduced in the context of the random potential problem by Derrida and Griffiths,[15] and offer the advantage that the position-space renormalization-group (RG) equations can be written down exactly. Such hierarchical lattices are frequently used as an approximation to regular lattices.[16] The ratio of the number of bonds between L_{n+1} and L_n is then used to define an effective dimension via $b = 2^{d-1}$.

In their initial work, Derrida and Griffiths[15] examined the directed paths subject to random bonds at *zero temperature*. Consider the evolution under RG of the probability distribution $\mathcal{P}_n(E)$ for the *minimum energy* of

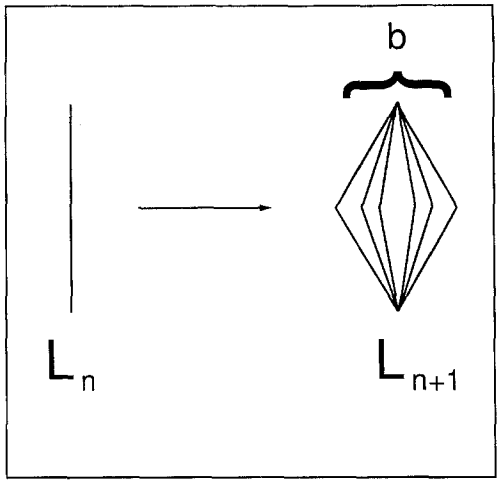

Fig. 2. Construction of the hierarchical lattice: At each successive level, each bond is replaced by $2b$ bonds as indicated.

all paths connecting the endpoints of the lattice. The energy of each branch of L_{n+1} is the sum of its two bonds, and obeys a probability distribution

$$\phi_n(E) = \int_{-\infty}^{\infty} dE' \, \mathscr{P}_n(E') \, \mathscr{P}_n(E' - E) \tag{2.1}$$

Selecting the minimum energy of the b branches then gives the RG equation

$$\mathscr{P}_{n+1}(E) = \frac{d}{dE}\left[\int_{E}^{\infty} dE' \, \phi_n(E')\right]^b \tag{2.2}$$

Numerical evolution of the above equation for $b = d = 2$ confirms that the width of $\mathscr{P}_n(E)$ grows as t^{ω} ($t = 2^n$) with $\omega \approx 0.30$, close to the value of 1/3 expected on a Euclidean two dimensional lattice. Halpin-Healy[17] has shown that the value of ω gradually decreases to zero as d is increased. Note that for $b = 1$ the problem reduces to simple addition of random variables. The central limit theorem than ensures that, as long as each random variable has a finite variance, $\mathscr{P}(E)$ is Gaussian with a width growing as t^{ω}, with $\omega = 1/2$. Although the value of ζ is not defined on such lattices, the exponent identity[18] $\omega = 2\zeta - 1$ can be used to obtain an estimate of this exponent.

Derrida and Griffiths[15] also constructed a $d = 1 + \varepsilon$ expansion for $\mathscr{P}(E)$, and obtained a value $\omega = 1/2 - \varepsilon K_2$, where K_2 is expressible in

terms of integrals of the error function, and has the numerical value $K_2 = 0.29782....$ Recently Roux *et al.*[19] considered the analytical form of the distribution for $d = 2$. They assumed that asymptotically the logarithm of $\mathscr{P}^*(E)$ is given by $-E_{\pm}^{\gamma}$ for $E \to \pm\infty$. Requiring these forms to be preserved under the RG equations (2.2) leads to the exponent identity

$$\gamma_+ = d\gamma_- \tag{2.3}$$

and a relationship to the fluctuation exponent given by

$$\omega = 1 - \frac{1}{\gamma_-} \tag{2.4}$$

They further assumed that the form of $\mathscr{P}^*(E)$ is preserved under a Laplace transformation, which leads to the additional identity

$$\gamma_+ = \frac{\gamma_-}{\gamma_- - 1} \tag{2.5}$$

Together these assumptions lead to $\omega = 1/3$ for $d = 2$, which is the correct value on a Euclidean lattice. They attribute the lower numerical estimate[15] as due to slow approach to the asymptotic limit. This result is very attractive, since its generalization to d dimensions leads to the exponent $\omega = 1/(d+1)$, conjectured to be exact for regular lattices.[20] Unfortunately, there is no justification for the assumption leading to Eq. (2.5), and, in fact, its conclusion disagrees with the aforementioned $1+\varepsilon$ expansion[15] at order ε.

For a diluted lattice, where the weights are 0 or 1, the above zero-temperature considerations are not immediately relevant, as all connected paths (in the absence of random bonds) have the same energy. The difference in the number of paths gives an entropic contribution that is important at finite temperature, where the appropriate quantity is the free energy. The recursion relation for the partition function at finite temperature, as originally obtained by Cook and Derrida,[21] is

$$Z' = Z_1 Z_2 + \cdots + Z_{2b-1} Z_{2b} \tag{2.6}$$

Clearly, after one step of the RG the weights Z are no longer restricted to 0 and 1, and hence the binary (0 or 1) distribution of randomness flows under the RG to the same problem with a distribution of positive weights (i.e., a random potential) in addition to the missing sites. That is, the 0–1 randomness problem is in the same universality class as that of a random potential on a percolating lattice. We expect this to be true for regular lattices as well.

The nonpercolating component can in fact be explicitly separated with a decomposition

$$p(Z) = p_0 \delta(Z) + (1 - p_0)\, \tilde{p}(Z) \tag{2.7}$$

From Eq. (2.6) one then readily derives a recursion relation for the probability of a bond being absent. It is interesting that this component decouples from the rest of the distribution and evolves as

$$p_0' = [1 - (1 - p_0)^2]^b \tag{2.8}$$

Equation (2.8) is precisely the position-space RG approximation for percolation on a hypercubic lattice (see ref. 22 for a review). The fixed point of this recursion relation determines p_c, the percolation threshold of the lattice. For $p > p_c$, p flows to 1, but in the process a random potential is generated, i.e., a nonuniform $\tilde{p}(Z)$. This confirms the expectation that small dilution of a uniform lattice has the same effect as random bonds.

It is the asymptotic behavior of $\tilde{p}(Z)$ that characterizes the nontrivial fluctuations of the free energy. As a first step we analyze the width of this distribution as characterized by the ratio of moments $w^2 \equiv (\langle Z^2 \rangle - \langle Z \rangle^2)/\langle Z \rangle^2$. The exact recursion relation for w obtained by considering moments of Eq. (2.6) is

$$\omega_{n+1}^2 = \frac{1}{bp_c} \{ w_n^4 + 2w_n^2 + (1 - p_c)[1 - (b-1)\, p_c] \} \tag{2.9}$$

The recursion relation for the pure problem can be regained by setting $p_c = 1$. In this case, it can be seen that for $b > 2$ there is a fixed point separating what appear to be phases of weak and strong randomness ($w \to 0$, or $w \to \infty$, respectively). This information is enough to determine the existence of a transition, but not its critical temperature or nonanalyticities.[21]

At $p = p_c$, simple analysis shows that Eq. (2.9) has no positive fixed point, and the width of the distribution flows to infinity for arbitrarily weak disorder in all dimensions at the percolation threshold. Thus there is no critical dimension for appearance of a weakly disordered phase on the percolating hierarchical lattice. We expect this result to also hold for Euclidean lattices. To proceed further and determine the nature of the strongly disordered phase we need to iterate Eq. (2.6) at the value of p_c obtained from the fixed point of Eq. (2.8). Since the distribution $\tilde{p}(Z)$ becomes very broad quite rapidly, it is more appropriate to consider $\tilde{p}[\ln(Z)]$. It is a simple matter to convert the recursion relation for $\tilde{p}(Z)$ to one for $\tilde{p}(\log Z)$. We studied the resulting integral equation numerically

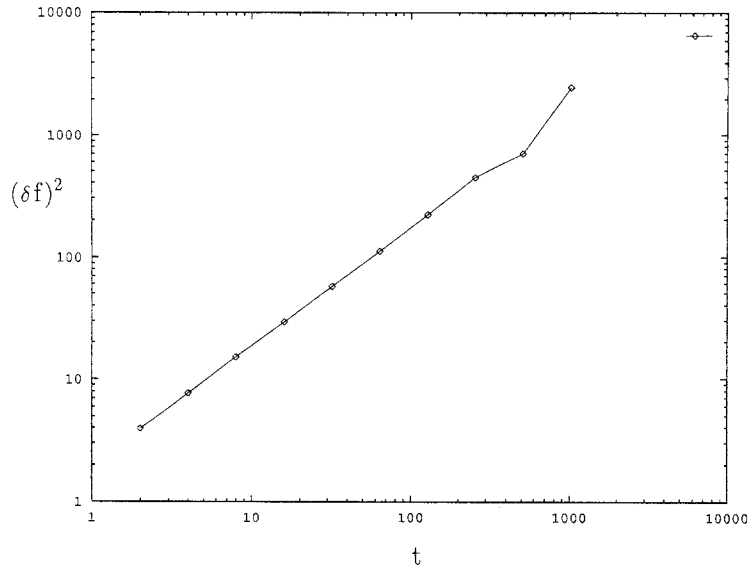

Fig. 3. Variance of ln Z versus t on the $b=2$ hierarchical lattice.

in $b=d=2$ by direct iteration. We found that the distribution becomes more and more narrow, with a width characterized by the exponent $\omega \approx 0.49 \pm 0.02$. (Fig. 3 is a plot of var ln Z versus t on a log-log scale.)

As discussed in the next section, we believe that up to possible logarithmic corrections, the exponent ω is exactly equal to 1/2 at percolation for all values of d. We attempted to test this hypothesis by a $1+\varepsilon$ expansion similar to that carried out at finite temperature by Cook and Derrida.[20] Unfortunately, we could not make much progress due to the nonanalyticity of such an expansion. This is already apparent when considering the fixed point of Eq. (2.8) which occurs at

$$p_c = 1 - \tfrac{1}{2} e^{-1/\varepsilon} \tag{2.10}$$

3. NUMERICAL SIMULATIONS IN $d=2$

We also performed simulations directly on the diluted square lattice depicted in Fig. 1 both for the simple percolation (i.e., 0–1 randomness) problem and with a random potential on top of the missing bonds. Unlike the hierarchical lattice problem, there is no obvious way to separate out the $Z=0$ portion of the probability distribution, since the different bonds do not remain statistically independent up renormalization. In order to calculate averages, however, one must choose some method of restriction

to the connected clusters. We chose what we believe to be the simplest possible criterion—that at least one connected path remains for the given time t. Computationally, we generated a configuration (which can be done concurrently with the evolution in t) and checked at each stage that the cluster retained this minimum connectivity. If it did not, we ignored this configuration and restarted the simulation. This becomes quite computationally intensive near the percolation threshold, as a large fraction of clusters must be thrown out before a connected cluster is found. The method of transfer matrices[18,1,4] can be used to evolve a weight $W(x, t)$ associated with each point on the lattice. In the case of 0–1 randomness, W simply measures the number of directed paths connecting the point (x, t) to the origin, while with random bonds present each path is weighted by the Boltzmann weight associated with the total energy of bonds crossed. The latter problem is mostly simulated at zero temperature, in which case it reduces to the problem of finding the path of lowest energy. We found identical results in both cases, although the random bond problem is found to converge faster.

We first confirmed that above the percolation threshold ($p_c = 0.6445$ for directed bond percolation on a square lattice[23]), the usual exponents[18,4] $\omega = 1/3$ and $\zeta = 2/3$ are recovered. This is illustrated in Figs. 4 and 5, which plot the transverse and free energy fluctuations for

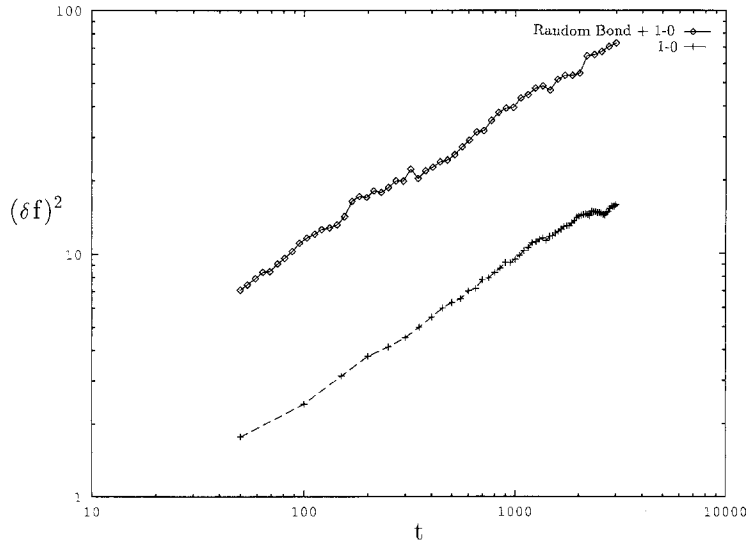

Fig. 4. Variance of $\ln Z$ versus t on the square lattice. The lower curve corresponds to empty/occupied bonds, while random energies are also added to occupied bonds for the results on the top curve.

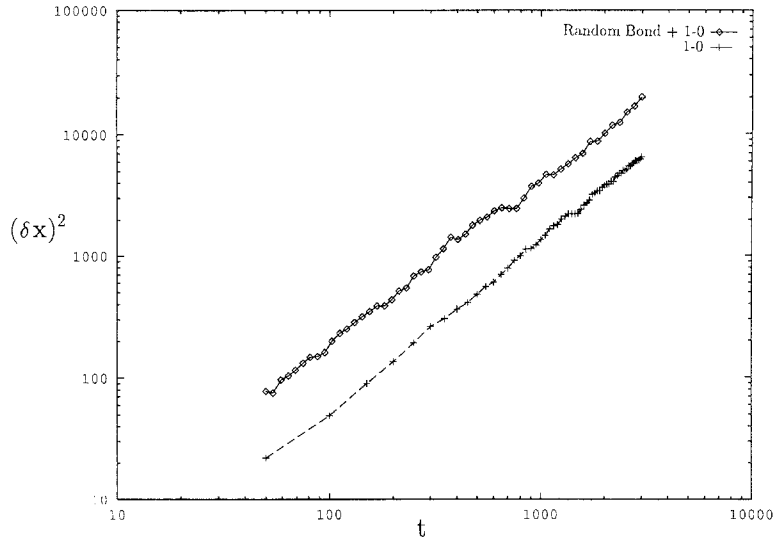

Fig. 5. Transverse fluctuations on the square lattice, for the two situations described in Fig. 4.

both types of distributions. The corresponding results at p_c are plotted in Figs. 6 and 7. Again the results confirm that the random-bond and simple percolation models satisfy similar asymptotic scaling. We find $\omega = 0.50 \pm 0.01$, as in the hierarchical lattices of the previous section. The transverse fluctuations cannot exceed the confines of the directed percolation clusters. These clusters are anisotropic, with correlation lengths that diverge as $\xi_{||} \sim |p_c - p|^{-\nu_{||}}$ and $\xi_\perp \sim |p_c - p|^{-\nu_\perp}$ on approaching percolation. Indeed, the observed value of $\zeta = 0.61 \pm 0.04$ is consistent with, if slightly lower than, the value[23] $\nu_\perp/\nu_{||} \approx 0.63$.

The above results strongly suggest that ω is exactly equal to one-half, which is the correct result in one dimension. A physical picture that supports this hypothesis is the following: The walker starts at the origin, and, not knowing that it is trapped on the percolation cluster, begins to wander with the two-dimensional exponent value of $\omega = 2/3$. However, since the confining directed percolation cluster itself only grows as $x \sim t^{0.63}$, it must eventually intersect the edges of the cluster. Thus, the walker explores as much of the space that as it is allowed, and $\zeta = \nu_\perp/\nu_{||}$. Because of this limited space for exploration (a cone of vanishing angle), the walk is essentially one-dimensional, and one expects the free energy fluctuation to exhibit the typical value $\omega = 1/2$.

Clearly the above argument is easily generalized to higher dimensions, and in general one expects a wandering exponent $\zeta = \min[\nu_\perp/\nu_{||}, \zeta_{RB}]$ at

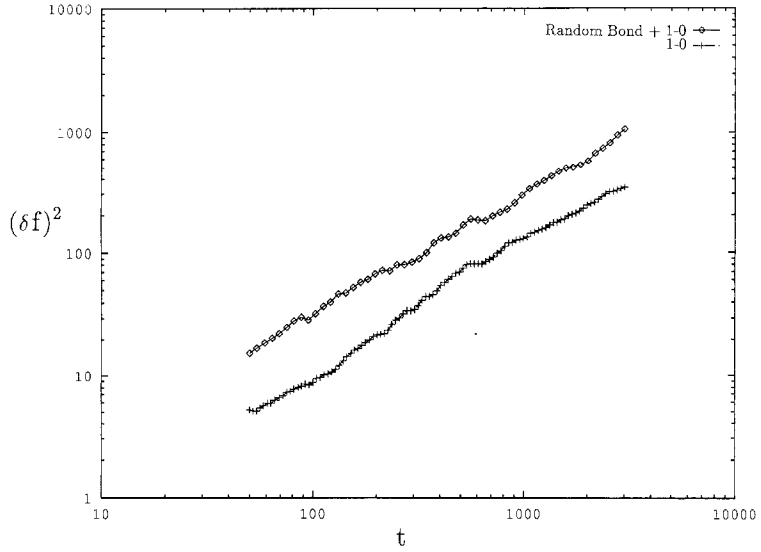

Fig. 6. As in Fig. 4, but at p_c.

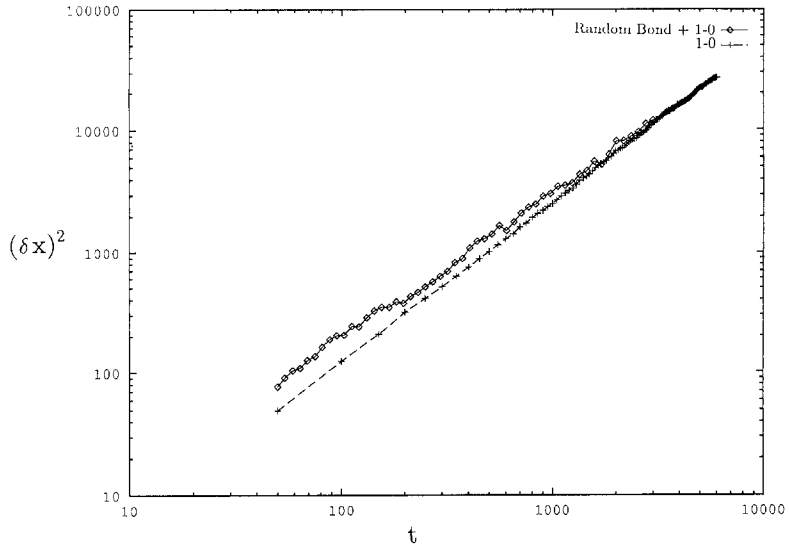

Fig. 7. As in Fig. 5, but at p_c.

the percolation threshold. Likewise, the free energy fluctuation exponent is expected to be $\omega = 1/2$ or ω_{RB} for the confined or free cases, respectively. Since the wandering exponent ζ_{RB} is believed to decrease in higher dimensions, one might imagine that above some dimension, ζ_{RB} will become smaller than the cluster width exponent $\nu_\perp/\nu_{||}$, and the walk will no longer be confined by the percolation cluster. However, above the critical dimension $d_c = 5$, mean-field theory holds for directed percolation,[23] and $\nu_\perp/\nu_{||}$ is known to be exactly 1/2. Since 1/2 is a lower bound for ζ_{RB} which is probably not reached at any finite critical dimension, the percolation cluster is expected always to be a "relevant" perturbation.

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